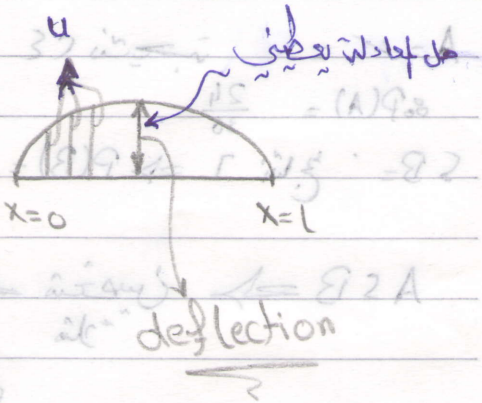
Solving the 1. dim. wave equation By Separating variables.



Given :- $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow c^2 = \frac{T}{\rho}$

→ get $u = u(x,t)$

Snap shot :-

1] Boundary conditions :- $\underbrace{u(0,t)}_{u@x=0} = 0$ & $\underbrace{u(l,t)}_{u@x=l} = 0$

2] Initial conditions :-

$u(x,0) = f(x)$ → deflection @ $t=0$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ → velocity @ $t=0$

Separation :- $u(x,t) = f(x) \cdot g(t)$

$u_{xx} = f'' \cdot g$ & $u_{tt} = f \cdot g'' \Rightarrow f \cdot g'' = c^2 f'' \cdot g \Rightarrow \frac{f''}{f} = \frac{g''}{c^2 g} = k$

1] $k > 0 \Rightarrow$ let $k = \mu^2 \therefore \frac{f''}{f} = \mu^2 \Rightarrow f'' - \mu^2 f = 0$

$\Rightarrow m = \pm\mu \Rightarrow f(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$ & $\frac{g''}{c^2 g} = \mu^2 \Rightarrow g'' - \mu^2 c^2 g = 0$

$\Rightarrow m = \pm c\mu \Rightarrow g(t) = C_3 e^{c\mu t} + C_4 e^{-c\mu t}$

? $k =$ zero

$f'' = 0 \Rightarrow f(x) = C_1 x + C_2$ & $g'' = 0 \Rightarrow g(t) = C_3 t + C_4$

3] $k < 0 \Rightarrow k = -\mu^2 \Rightarrow \therefore m = \pm i\mu \Rightarrow f(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$

$$\Rightarrow \frac{g''}{c^2 g} = -\mu^2 \Rightarrow g'' + c^2\mu^2 g = 0 \Rightarrow m = \pm c\mu i$$

$$\therefore g(t) = C_3\cos(c\mu t) + C_4\sin(c\mu t)$$

ونحدد K (1 or 2 or 3) من خلال ال int. cond.

How to find the required case for k:

① for $k>0$, $f(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$, $g(t) = c_3 e^{\mu ct} + c_4 e^{-\mu ct}$

B.C. (Bound. Cond.) $u(0,t) = 0 \Rightarrow f(0).g(t) = 0 \Rightarrow g(t) \neq 0 \quad \therefore f(0) = 0 \Rightarrow c_1 + c_2 = 0$

$, u(l,t) = 0 \Rightarrow f(l).g(t) = 0 \Rightarrow f(l) = 0 \Rightarrow c_1 e^{\mu} + c_2 e^{-\mu} = 0$; $c_1 + c_2 = 0$

$\Rightarrow f(x) = 0 \Rightarrow u(x,t) = 0 \equiv$ الحل التافه $\Rightarrow \therefore k>0$ ✗

② $k=0 \Rightarrow f(x) = c_1 x + c_2$, $g(t) = c_3 t + c_4$ $\therefore f(0) = 0 \Rightarrow c_2 = 0$

$\therefore f(x) = c_1 x$, $f(l) = 0 \Rightarrow c_1 l = 0 \Rightarrow c_1 = 0$ ← حل تافه تاني! ✗

$\therefore$ The required case is $k<0$

والأعم في حالة ال wave equation فقط. انّ $k<0$

$$\therefore f(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$$

$$, g(t) = c_3\cos(c\mu t) + c_4\sin(c\mu t)$$

$\therefore f(0) = 0 \Rightarrow c_1 = 0 \Rightarrow f(x) = c_2\sin(\mu x)$, $\because f(l) = 0 \Rightarrow c_2\sin(\mu l) = 0$

$, c_2 \neq 0$ (لأن $f(x) \neq 0$) , $\therefore \sin(\mu l) = 0 \Rightarrow \mu l = n\pi \Rightarrow \mu = \frac{n\pi}{l}$

$$\therefore f(x) = c_2 \sin\frac{n\pi}{l}x \quad , u = f(x).g(t)$$

Initial conditions: $u(x,0) = h(x)$, $u_t(x,0) = q(x)$

let $u(x,t) = \left[A\cos(c\mu t) + B\sin(c\mu t)\right]\sin\frac{n\pi}{l}x$

where $A = C_2C_3$, $B = C_2C_4$

$$\Rightarrow u_n(x,t) = \left(A\cos\frac{cn\pi}{l}t + B\sin\frac{cn\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

Superposition principle:- $u_1, u_2, u_3, \dots\dots u_k$

$$u = u_1 + u_2 + u_3 + \dots\dots + u_k$$

The general sol. of the wave eqn. is:-

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{cn\pi}{l}t + B_n\sin\frac{cn\pi}{l}t\right)\sin\frac{n\pi}{l}$$

Int. cond.:- $u(x,0) = h(x) \Rightarrow h(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x$

ومفاجأة ان دي هي فورير في حالة الدالة الفردية !!!

$$\to f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{l} \qquad \therefore A_n = \frac{2}{l}\int_0^l h(x)\sin\frac{n\pi x}{l}\,dx$$

* الإثبات اللي فات ده مش مطلوب لكن المطلوب ← البرهان في wave

Sub.

$$\therefore A_n = \frac{2}{l}\int_0^l h(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

, $u_t(x,0) = g(x) \Rightarrow$ velocity at $t = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-\frac{cn\pi}{l}A_n\sin\frac{cn\pi}{l}t + \frac{cn\pi}{l}B_n\cos\frac{cn\pi}{l}t\right]\cdot\sin\frac{n\pi}{l}x$$

at $t = 0 \Rightarrow$ $$B_n = \frac{2}{cn\pi}\int_0^l g(x)\cdot\sin\frac{n\pi}{l}x\,dx$$

* لو السرعة = 0 يعني بدأ من سكون ⇐ $g(x) = 0$ ⇐ $B_n = 0$

Ex:- Solve $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, given $u(0,t) = 0$, $u(l,t) = 0$

, $u(x,0) = \frac{1}{4}x(L - x) = h(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \Rightarrow B_n = 0$

بالتجزيء

$$\therefore A_n = \frac{2}{l}\int_0^l \frac{1}{4}x(l - x)\sin\frac{n\pi}{l}x\,dx$$

$$\Rightarrow A_n = \frac{l^2}{n^3\pi^3}\left(1 - (-1)^n\right)$$